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CS109

Handout #12
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Poisson and More Discrete Distributions

Poisson random variables will be the third main discrete distribution that we expect you to know well. After introducing Poisson, we will quickly introduce three more. I want you to be comfortable with being told the semantics of a distribution, given the key formulas (for expectation, variance and PMF) and then using it.

Binomial in the Limit

Recall example of sending bit string over network. In our last class we used a binomial random variable to represent the number of bits corrupted out of four with a high corruption probability (each bit had independent probability of corruption $p = 0.1$). That example was relevant to sending data to space craft, but for earthly applications like HTML data, voice or video, bit streams are much longer (length $\approx 10^4$) and the probability of corruption of a particular bit is very small ($p \approx 10^{-6}$). Extreme n and p values arise in many cases: # visitors to a website, #server crashes in a giant data center.

Unfortunately X $\sim$ Bin(10^4, 10^{-6}) is unwieldy to compute. However when values get that extreme we can make approximations that are accurate and make computation feasible. Recall the Binomial distribution. First define $\lambda = np$. We can rewrite the Binomial PMF as follows:

$$\begin{aligned} P(X = i) &= \frac{n!}{i!(n-1)!}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i} \\ &= \frac{n(n-1)\dots(n-i-1)}{n^i}\frac{\lambda^i}{i!}\frac{(1-\lambda/n)^n}{(1-\lambda/n)^i} \end{aligned}$$

This equation can be made simpler by observing how some of these equations evaluate when n is sufficiently large and p is sufficiently small. The following equations hold:

$$\frac{n(n-1)\dots(n-i-1)}{n^i} \approx 1$$
$$(1-\lambda/n)^n \approx e^{-\lambda}$$
$$(1-\lambda/n)^i \approx 1$$

This reduces our original equation to:

$$P(X = i) = \frac{\lambda^i}{i!}e^{-\lambda}$$

This simplification turns out to be so useful, that in extreme values of n and p we call the approximated Binomial its own random variable type: the Poisson Random Variable.

Poisson Random Variable

A Poisson random variable approximates Binomial where n is large, p is small, and $\lambda = np$ is “moderate”. Interestingly, to calculate the things we care about (PMF, expectation, variance) we no longer need to know n and p. We only need to provide λ which we call the rate.

There are different interpretations of ”moderate”. The accepted ranges are $n > 20$ and $p < 0.05$ or $n > 100$ and $p < 0.1$.

Here are the key formulas you need to know for Poisson. If $Y \sim Poi(\lambda)$:

$$P(Y = i) = \frac{\lambda^i}{i!} e^{-\lambda}$$
$$E[Y] = \lambda$$
$$Var(Y) = \lambda$$

Example

Let's say you want to send a bit string of length $n = 10^4$ where each bit is independently corrupted with $p = 10^{-6}$. What is the probability that the message will arrive uncorrupted? You can solve this using a Poisson with $\lambda = np = 10^4 10^{-6} = 0.01$. Let $X \sim Poi(0.01)$ be the number of corrupted bits. Using the PMF for Poisson:

$$\begin{aligned} P(X = 0) &= \frac{\lambda^i}{i!} e^{-\lambda} \\ &= \frac{0.01^0}{0!} e^{-0.01} \\ &\sim 0.9900498 \end{aligned}$$

We could have also modelled X as a binomial such that $X \sim Bin(10^4, 10^{-6})$. That would have been computationally harder to compute but would have resulted in the same number (up to the millionth decimal).

Geometric Random Variable

X is Geometric Random Variable: $X \sim Geo(p)$ if X is number of independent trials until first success and p is probability of success on each trial. Here are the key formulas you need to know. If $X \sim Geo(p)$:

$$P(X = n) = (1 - p)^{n-1} p$$
$$E[X] = 1/p$$
$$Var(X) = (1 - p)/p^2$$

Negative Binomial Random Variable

X is Negative Binomial: $X \sim NegBin(r, p)$ if X is number of independent trials until r successes and p is probability of success on each trial. Here are the key formulas you need to know. If $X \sim NegBin(p)$:

$$P(X = n) = \binom{n-1}{r-1} p^r (1 - p)^{n-r} \text{ where } r \leq n$$
$$E[X] = r/p$$
$$Var(X) = r(1 - p)/p^2$$

Hypergeometric Random Variable

X is Hypergeometric: $X \sim HypG(n, N, m)$ if X is the number of white balls drawn from an urn with N balls: $(N - m)$ black and m white:

$$P(X = i) = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}} \text{ where } i \geq 0$$
$$E[X] = n(m/N)$$
$$Var(X) = \frac{nm(N - n)(N - m)}{N^2(N - 1)}$$